Lecture 23 04/16/2018

Waveguides (Cont'd)

(b) TM modes. In this case, $\vec{B}$ is transverse everywhere, $B_z=0$, while $E_z \neq 0$. From Maxwell equations, we have:

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B} = i\omega \vec{B}_T \Rightarrow \left(\hat{z}\frac{\partial}{\partial z} + \vec{\nabla}_T\right) \times \vec{E} = i\omega \vec{B}_T \Rightarrow ik\,\hat{z}\times\vec{E} + \vec{\nabla}_T \times \vec{E}$$

$$= i\omega \vec{B}_T \Rightarrow \hat{z}\times\left[ik(\hat{z}\times\vec{E}) + \vec{\nabla}_T\times\vec{E}\right] = i\omega\,\hat{z}\times\vec{B}_T$$

But:

$$\hat{z}\times(\hat{z}\times\vec{E}) = \hat{z}\times(\hat{z}\times\vec{E}_T) = (\overset{0}{\cancel{\hat{z}\cdot\vec{E}_T}})\hat{z} - \vec{E}_T$$

And:

$$\hat{z}\times(\vec{\nabla}_T\times\vec{E}) = \vec{\nabla}_T(\underbrace{\hat{z}\cdot\vec{E}}_{E_z}) - (\overset{0}{\cancel{\hat{z}\cdot\vec{\nabla}_T}})\vec{E}$$

Thus:

$$\boxed{i\omega\,\hat{z}\times\vec{B}_T + ik\vec{E}_T = \vec{\nabla}_T E_z}$$

Also:

$$\hat{z}\cdot\left[ik(\hat{z}\times\vec{E}) + \vec{\nabla}_T\times\vec{E}\right] = \hat{z}\cdot i\omega\,(\overset{0}{\cancel{\hat{z}\times\vec{B}_T}}) \Rightarrow \hat{z}\cdot(\vec{\nabla}_T\times\vec{E}) = 0$$

$$\Downarrow$$

$$\hat{z}\cdot(\hat{z}\times\vec{E}) = 0$$

$$\Rightarrow \hat{z}\cdot(\vec{\nabla}_T \times \vec{E}_T) = 0 \Rightarrow \boxed{\vec{\nabla}_T \times \vec{E}_T = 0}$$

In addition, we have:

$$\vec{\nabla} \times \vec{B} = -i\nu\varepsilon\omega \vec{E} \Rightarrow -i\nu\varepsilon\omega\,\hat{z}\times \vec{E}_T + ik\vec{B}_T = \vec{\nabla}_T B_z = 0$$

Hence:

$$\boxed{\vec{B}_T = \frac{\nu\varepsilon\omega}{k}\,\hat{z}\times\vec{E}_T} \qquad *$$

Also:

$$\hat{z}\cdot(\vec{\nabla}_T \times \vec{B}_T) = -i\nu\varepsilon\omega E_z$$

The first and third boxed equations yield:

$$ik\vec{E}_T + i\omega^2\frac{\nu\varepsilon}{k}\underbrace{\hat{z}\times(\hat{z}\times\vec{E}_T)}_{-\vec{E}_T} = \vec{\nabla}_T E_z \Rightarrow ik\left(1 - \frac{\nu\varepsilon\omega^2}{k^2}\right)\vec{E}_T = \vec{\nabla}_T E_z$$

$$\Rightarrow \boxed{\vec{E}_T = \frac{ik}{\nu\varepsilon\omega^2 - k^2}\vec{\nabla}_T E_z} \qquad **$$

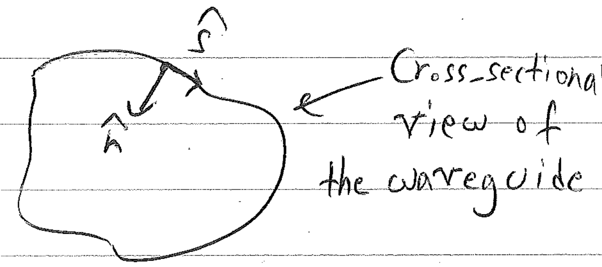
We see from $*$ and $**$ that $\vec{E}_T$ and $\vec{B}_T$ can be expressed in terms of E_z. E_z itself obeys the wave equation:

$$(\nabla^2 + \omega^2\nu\varepsilon)E_z = 0 \Rightarrow [\nabla_T^2 + (\omega^2\nu\varepsilon - k^2)]E_z = 0$$

This is basically a Helmholtz equation:

$$(\nabla_T^2 + \gamma^2) E_z = 0 \quad , \quad \gamma^2 \equiv \omega^2 \mu \epsilon - k^2$$

We now need to impose the boundary condition. At a conducting wall the tangential component of $\vec{E}$ and the normal component of $\vec{B}$ must vanish. Therefore:



$$E_z = 0$$

$$\vec{E}_T \cdot \hat{s} = 0 \Rightarrow \frac{\partial E_z}{\partial s} = 0$$

$$\vec{B}_T \cdot \hat{n} = 0 \Rightarrow \frac{\partial E_z}{\partial s} = 0$$

We note that $E_z = 0$ everywhere on the wall implies that $\frac{\partial E_z}{\partial s} = 0$.

Therefore, the boundary conditions are satisfied if $\underline{\underline{E_z = 0}}$.

(c) TE modes. In this case, $\vec{E}$ is transverse everywhere, $E_z = 0$, while $B_z \neq 0$. We have:

$$\boxed{\vec{E}_T = -\frac{\omega}{k} \hat{z} \times \vec{B}_T}$$

And:

$$ik\left(\vec{B}_T - \frac{\mu\epsilon\omega}{k}\hat{z}\times\vec{E}_T\right) = \vec{\nabla}_T B_z \Rightarrow ik\left[\vec{B}_T + \frac{\mu\epsilon\omega^2}{k^2}\underbrace{\hat{z}\times(\hat{z}\times\vec{B}_T)}_{-\vec{B}_T}\right] = \vec{\nabla}_T B_z$$

$$\Rightarrow \boxed{\vec{B}_T = \frac{ik}{\gamma^2}\vec{\nabla}_T B_z}$$

B_z obeys the wave equation:

$$(\nabla_T^2 + \gamma^2)B_z = 0$$

The boundary conditions at the wall are:

$$\vec{B}_T \cdot \hat{n} = 0 \Rightarrow \frac{\partial B_z}{\partial n} = 0$$

$$\vec{E}_T \cdot \hat{s} = 0 \Rightarrow \frac{\partial B_z}{\partial n} = 0$$

They are both satisfied if $\underline{\frac{\partial B_z}{\partial n} = 0}$.

We see that for both the TM and TE modes we have to solve the Helmholtz equation:

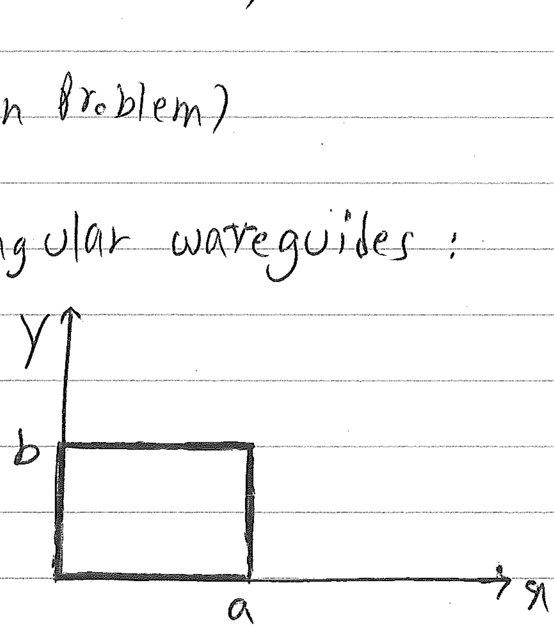
$$(\nabla_T^2 + \gamma^2)\psi = 0$$

Where:

TM: $\psi = E_z$, $\psi|_S = 0$ (Dirichlet Problem)

TE: $\psi = B_z$, $\frac{\partial \psi}{\partial n}\Big|_S = 0$ (Neumann Problem)

For example, let us consider rectangular waveguides:



For the TM modes, we have:

$$\psi_{mn}(x,y) = A \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right)$$

so that $\psi_{mn} = 0$ at $x = 0, a$ and $y = 0, b$.

This implies that:

$$\gamma_{mn} = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \qquad m, n = 1, 2, \ldots$$

We note that m, n are both <u>non-zero</u> integers as $m = 0$ or $n = 0$ results in $E_z = 0$ (hence TEM mode, which cannot be supported by a rectangular wall). Therefore:

$$\boxed{E_z = A \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) e^{i\sqrt{\omega^2 \mu \epsilon - \gamma_{mn}^2}\, z - i\omega t}}$$

$\vec{E}_T$ and $\vec{B}_T$ are then determined according to:

$$\boxed{\vec{E}_T = \frac{ik}{\gamma_{mn}^2} \vec{\nabla}_T E_z \, , \qquad \vec{B}_T = \frac{\mu \epsilon \omega}{k} \hat{z} \times \vec{E}_T}$$

For the TE modes, we have:

$$\psi_{mn}(x,y) = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), \quad \gamma_{mn} = \pi\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}} \quad m,n = 0,1,2,\dots$$

(both m, n cannot be zero)

Hence:

$$\boxed{B_z = A\cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) e^{i\sqrt{\omega^2\mu\epsilon - \gamma_{mn}^2}\, z - i\omega t}}$$

We note that only modes with $k^2 > 0$ can propagate in the waveguide.

This requires that:

$$\omega^2 > \frac{\gamma_{mn}^2}{\mu\epsilon}$$

For ω smaller than the minimum value of $\frac{\gamma_{mn}}{\sqrt{\mu\epsilon}}$ (called the "cut-off frequency") there can be no propagating mode. The cut-off frequency for the TM modes correspond to $m=n=1$, while that for the TE modes correspond to $m=1$ and $n=0$ ($m=0$ and $n=1$) if $a>b$ ($a<b$).

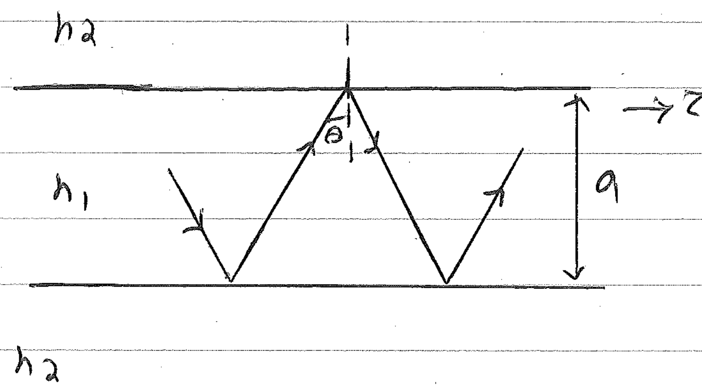
Dielectric Waveguides

These typically consist of an inner layer of dielectric material where the radiation is confined, and one or more outer layers of material with lower index of refraction. The fields therefore decay exponentially away from the interface.

Propagation of waves may be understood in terms of total internal reflection at the interface of the inner and outer cores. For example, Consider a slab waveguide as follows;

Here, we assume that $n_1 > n_2$.

At each reflection, the wave picks up a phase shift



of ϕ_{12}. A round trip from the bottom interface to the top and back then results in a total phase shift:

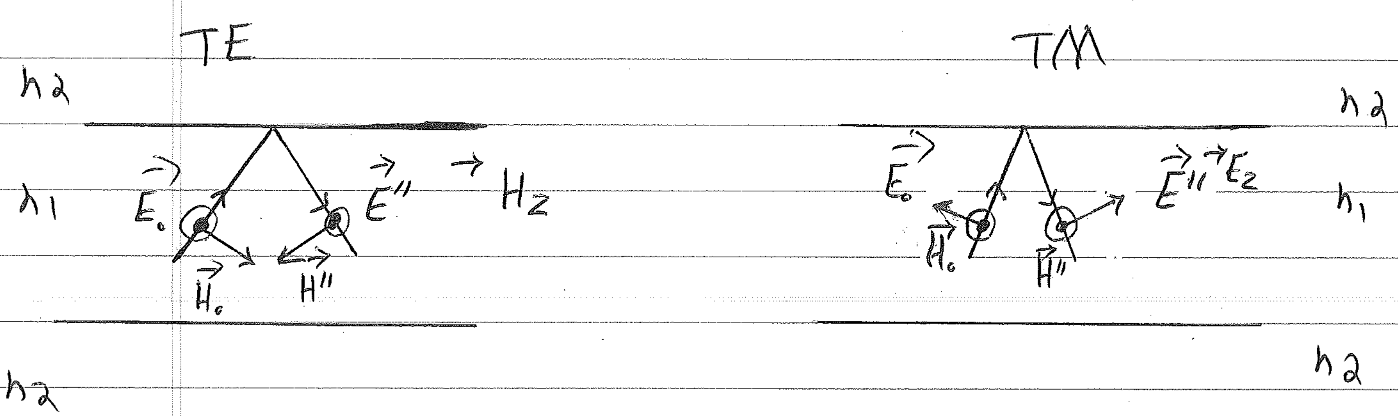
$$2k\cos\theta\, a + 2\phi_{12} = 2\frac{\omega}{c} n_1 \cos\theta\, a + 2\phi_{12}$$

In order for the wave to propagate in the z direction, we need:

$$2\frac{\omega}{c} n_1 \cos\theta\, a + 2\phi_{12} = 2p\pi \quad (p \text{ an integer})$$

As well as $\theta > \theta_{crit}$ (in order for total internal reflection to occur). The two conditions together result in a discrete set of θ's, corresponding to discrete values of p. For a given ω, there is a maximum value of p, denoted by p_{max}, for which $\theta \geq \theta_{crit}$ has its minimum. The lower p compared to p_{max} is, the more confined the propagating mode in the inner layer will be.

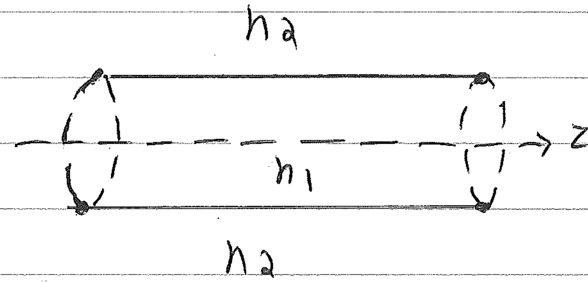
The configurations for the TE and TM modes are as follows in this case:



No TEM modes are possible for a slab waveguide.

The above considerations are not easily implementable for circular cross-sections ("optical fibers").

In this case, the wave equation in the two regions becomes:



$$\left(\nabla^2 + \frac{\omega^2 n_1^2}{c^2}\right)\begin{cases}\vec{E} \\ \vec{H}\end{cases} = 0 \quad , \quad \left(\nabla^2 + \frac{\omega^2 n_2^2}{c^2}\right)\begin{cases}\vec{E} \\ \vec{H}\end{cases} = 0$$

Solutions of the form $e^{i(kz-\omega t)}$ in both regions with the same k are possible:

$$\left[\nabla_T^2 + \left(\frac{\omega^2 n_1^2}{c^2} - k^2\right)\right]\begin{cases}\vec{E} \\ \vec{H}\end{cases} = 0 \quad , \quad \left[\nabla_T^2 + \left(\frac{\omega^2 n_2^2}{c^2} - k^2\right)\right]\begin{cases}\vec{E} \\ \vec{H}\end{cases} = 0$$

Propagation in region 1 and evanescent waves in region 2 can be arranged if $\frac{\omega n_2}{c} < k < \frac{\omega n_1}{c}$. For general field dependences, neither of E_z or H_z can be made to vanish. Hence, in general, TE and TM modes are not possible. The exception is when the fields have no azimuthal dependence.